

An Introduction to Alternating Direction Method of Multipliers (ADMM)

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Distributed Optimization and Statistical Learning via the Alternating Direction Method of Multipliers

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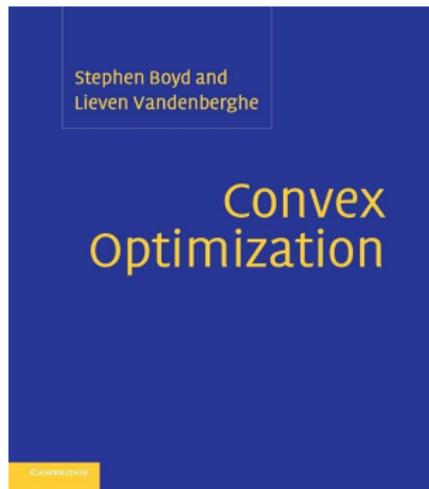
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- 1 Lagrangian Duality
- 2 From Dual Asent to ADMM
- 3 Why ADMM
- 4 ADMM in CliPP

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Primary Problem

$$\begin{aligned} \min_x & f(x) \\ \text{s.t.} & h_i(x) \leq 0, i \in \{1, \dots, I\} \\ & \ell_j(x) = 0, j \in \{1, \dots, J\} \end{aligned}$$

- The objective function $f(x)$ usually has poor properties (non-convex or no Lipschitz continuity).
- The feasible region is not always a convex set.

Lagrangian of the Primary Problem

$$L(x, \lambda, \nu) = f(x) + \sum_{i=1}^I \lambda_i h_i(x) + \sum_{j=1}^J \nu_j \ell_j(x),$$

where λ, ν are Lagrangian multipliers.

Dual function

$$g(\lambda, \nu) = \inf_{x \in \text{dom} f} (f(x) + \sum_{i=1}^I \lambda_i h_i(x) + \sum_{j=1}^J \nu_j \ell_j(x)).$$

Note that, $g(\lambda, \nu)$ is concave w.r.t λ, ν , and the above infimum is taken over the domain of f .

A brief introduction to Lagrangian Duality

Primary Problem

$$\min_x f(x)$$

$$\text{s.t. } h_i(x) \leq 0, i \in \{1, \dots, I\}$$

$$\ell_j(x) = 0, j \in \{1, \dots, J\}$$

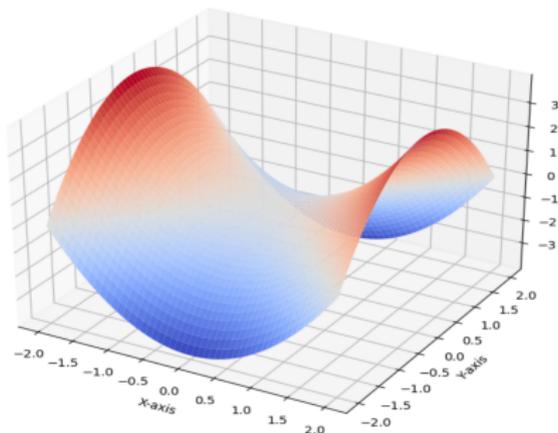
The dual problem is much easier to solve, as it is a convex optimization with linear constraints.

Dual Problem

$$\max_{\lambda, \nu} g(\lambda, \nu)$$

$$\text{s.t. } \lambda \geq 0.$$

Saddle Point Visualization: $f(x, y) = x^2 - y^2$



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Dual Ascent

Consider an easier problem $\min_x f(x)$ s.t. $Ax = b$.

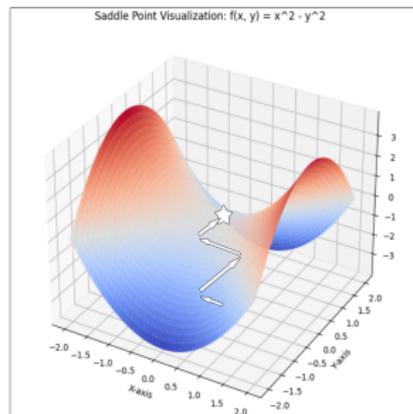
Its dual function is

$$g(y) = \inf_{x \in \text{dom}f} f(x) + y^T (Ax - b).$$

The gradient of $g(y)$ is $\frac{\partial g}{\partial y} = Ax^* - b$

The dual ascent algorithm is

- Initialize dual guess $y^{(0)}$
- repeat for $k = 1, 2, 3, \dots$
 - $x^{(k)} = \arg \min_{x \in \text{dom}f} f(x) + (y^{(k-1)})^T Ax$
 - $y^{(k)} = y^{(k-1)} + t_k (Ax^{(k)} - b)$



Augmented Lagrangian method a.k.a method of multipliers

The disadvantage of dual ascent is that it requires strong conditions to ensure convergence, i.e., convexity and Lipschitz continuity, etc. (It can be considered similar to the gradient descent algorithm.)

We propose a new primal problem

$$\begin{aligned} \min_x \quad & f(x) + \frac{\rho}{2} \|Ax - b\|_2^2 \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

The benefit that comes with the quadratic penalty term is that we can always adjust ρ such that the objective function is convex (under mild assumptions), as long as matrix A has full column rank.

$$\begin{aligned} \min_x \quad & f(x) + \frac{\rho}{2} \|Ax - b\|_2^2 \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

- Initialize dual guess $y^{(0)}$
- repeat for $k = 1, 2, 3, \dots$
 - $x^{(k)} = \arg \min_{x \in \text{dom} f} f(x) + (y^{(k-1)})^T Ax + \frac{\rho}{2} \|Ax - b\|_2^2$
 - $y^{(k)} = y^{(k-1)} + t_k (Ax^{(k)} - b)$

Alternating Direction Method of Multipliers (ADMM)

Consider the problem

$$\begin{aligned} \min_{x,z} \quad & f(x) + g(z) \\ \text{s.t.} \quad & Ax + Bz = c \end{aligned}$$

The augmented Lagrangian is

$$L_\rho(x, z, u) = f(x) + g(z) + u^T (Ax + Bz - c) + \frac{\rho}{2} \|Ax + Bz - c\|_2^2$$

The algorithm is

- repeat for $k = 1, 2, 3, \dots$
 - $x^{(k)} = \arg \min_x L_\rho(x, z^{(k-1)}, u^{(k-1)})$
 - $z^{(k)} = \arg \min_z L_\rho(x^{(k)}, z, u^{(k-1)})$
 - $u^{(k)} = u^{(k-1)} + t_k (Ax^{(k)} + Bz^{(k)} - c)$

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Divide and Conquer

It can split a large problem into a series of subproblems. Usually, each subproblem has a closed form solution. Consider problem

$$\min_x f(x) + g(Ax)$$

We can transform it into

$$\min_{x,z} f(x) + g(z), \text{ s.t. } Ax - z = 0.$$

$$L_\rho(x, z, u) = f(x) + g(z) + u^T(Ax + Bz - c) + \frac{\rho}{2}\|Ax + Bz - c\|_2^2$$

• repeat for $k = 1, 2, 3, \dots$

- $x^{(k)} = \arg \min_x g(z) + (u^{(k-1)})^T Ax + \frac{\rho}{2}\|Ax + Bz^{(k-1)} - c\|_2^2$
- $z^{(k)} = \arg \min_z f(x) + (u^{(k-1)})^T Bz + \frac{\rho}{2}\|Ax^{(k)} + Bz - c\|_2^2$
- $u^{(k)} = u^{(k-1)} + t_k(Ax^{(k)} + Bz^{(k)} - c)$

Distributed Optimization

Given $y \in \mathbb{R}^n$, $x \in \mathbb{R}^{n \times p}$, we have the group lasso problem as

$$\min_{\beta} \frac{1}{2} \|y - X\beta\|_2^2 + \lambda \sum_{g=1}^G c_g \|\beta_g\|_2$$

Rewrite as

$$\min_{\alpha, \beta} \frac{1}{2} \|y - X\beta\|_2^2 + \lambda \sum_{g=1}^G c_g \|\alpha_g\|_2, \text{ s.t. } \beta - \alpha = 0.$$

ADMM steps:

- repeat for $k = 1, 2, 3, \dots$
 - $\beta^{(k)} = (X^T X + \rho I)^{-1} (X^T y + \rho(\alpha^{(k-1)} - \omega^{(k-1)}))$
 - for $g = 1, \dots, G$ do in parallel
 - $\alpha_g^{(k)} = R_{c_g \lambda / \rho}(\beta^{(k)} + \omega_g^{(k-1)})$
 - $\omega^{(k)} = \omega^{(k-1)} + \beta^{(k)} - \alpha^{(k)}$

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The problem, eq.(8), in CliPP is

$$\min_{\omega} -\ell(\omega) + \sum_{1 \leq i < j \leq s} \rho_{\lambda}(|\omega_i - \omega_j|).$$

Define $\eta_{ij} = \omega_i - \omega_j$, we can rewrite it as

$$\min_{\omega, \eta} -\ell(\omega) + \sum_{1 \leq i < j \leq s} \rho_{\lambda}(|\eta_{ij}|).$$

Then the augmented Lagrangian is

$$\begin{aligned} L(\omega, \eta, \tau, \lambda) = & -\ell(\omega) + \sum_{1 \leq i < j \leq s} \rho_{\lambda}(|\eta_{ij}|) + \frac{\alpha}{2} \sum_{1 \leq i < j \leq s} (\eta_{ij} - \omega_i - \omega_j)^2 \\ & - \sum_{1 \leq i < j \leq s} \tau_{ij}(\eta_{ij} - \omega_i - \omega_j). \end{aligned}$$

- repeat for $k = 1, 2, 3, \dots$
 - eq.(S3) $\boldsymbol{\omega}^{(k)} = (\mathbf{B}^T \mathbf{B} + \alpha \boldsymbol{\Delta}^T \boldsymbol{\Delta})^{-1} [\alpha \boldsymbol{\Delta}^T (\boldsymbol{\eta}^{(k-1)} - \boldsymbol{\tau}^{(k-1)}) - \mathbf{B}^T \mathbf{A}]$
 - eq.(S4) $\eta_{ij}^{(k)} = \arg \min_{\eta_{ij}} \frac{\alpha}{2} (\delta_{ij} - \eta_{ij})^2 + p_\lambda (|\eta_{ij}|)$
 - eq.(S5) $\boldsymbol{\tau}^{(k)} = \boldsymbol{\tau}^{(k-1)} - \alpha (\boldsymbol{\Delta} \boldsymbol{\omega}^{(k)} - \boldsymbol{\eta}^{(k)})$

<https://github.com/wwylab/CliPP/blob/master/src/kernel.cpp>

- Update ω
- Update η
- Update τ

```
254     linear = DELTA * (alpha * eta_old + tau_new) - B.cwiseProduct(A);
255
256     Minv = 1.0 / ((B.cwiseProduct(B)).array() + double(no_mutation) * alpha);
257     Minv_diag = Minv.asDiagonal();
258
259     trace_g = -alpha * Minv.sum();
260     if(!isnan(trace_g)){
261         std::cout << "Lambda: " << Lambda << "\titeration: " << k << "\t\titan" << std::endl;
262         return -1;
263     }
264
265
266     Minv_outer = Minv * (Minv.transpose());
267
268     inverted = Minv_diag.array() - 1.0 / (1.0 + trace_g) * (-alpha) * Minv_outer.array();
269     w_new = inverted * linear;
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Thank you!
Questions?



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